PHYSICAL UNITARITY IN THE BRST APPROACH

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Received 4 August 1988

The conditions of unitarity in the positive subspace of an indefinite metric theory are given. BRST quantization is considered as the most important example.

The covariant formulation of the most physically interesting models requires the introduction of an indefinite metric space. One of the most popular examples is given by the BRST quantization of gauge theories [1–5]. In this approach the S-matrix is defined in a large space including ghost states, and the physical subspace is separated by the condition

$$Q|\phi\rangle=0\,, \tag{1}$$

where Q is the BRS charge. The operator Q commutes with the hamiltonian, therefore the S-matrix is obviously unitary in this subspace. However, it is by no means evident that the states (1) are really "physical", i.e. have a non-negative norm. For the Yang–Mills theory this fact was proven in refs. [1,2]. There exists also a proof of the equivalence of the BRST quantization and the Dirac quantization for the gauge theories with independent constraints and closed algebra [6,5]. However, in the general case when the constraints are dependent and the algebra may be open the proof of the positivity of the physical states norm was absent.

In this paper we formulate the conditions providing the unitarity of the S-matrix in the positive norm subspace. No assumptions about the gauge group and the existence of a unitary gauge is needed. In fact our proof works equally well when there is no gauge invariance at all.

We consider a theory described by the Lorentz invariant non-degenerate lagrangian $\mathscr{L}(A_i, c_j^\alpha, \bar{c}_k^\alpha)$. Here c_j^α, $\bar{c}_k^\alpha$ are ghost and antighost fields, and the index α denotes ghost number. All the fields are hermitian. We assume also that the free lagrangian is quadratic.

The corresponding creation and annihilation operators satisfy the standard (anti)commutation relations

$$[c_j^\alpha(k), \bar{c}_l^\beta(k')]_\pm=\delta^{\alpha\beta}\delta(k-k')\,,$$
$$g_{jl}^\alpha=\mathrm{diag}\{+1,\ldots,+1,-1,\ldots,-1\}\,. \tag{2}$$

Analogous relations are valid for the operators $a_i(k)$, $a_j(k')$. The fields with ghost numbers differing by 1 have an opposite Grassmann parity.

There exists an operator $Q(a_i, a_j^*, c_l^\alpha, c_k^{\alpha*}, \bar{c}_m^\alpha, c_n^{\alpha*})$ with the following properties:

(1) The operator Q allows a representation

$$Q=Q_0+\mathrm{O}(g)\,, \tag{3}$$

where Q_0 is quadratic in the fields and $\mathrm{O}(g)$ denotes higher order terms vanishing when the interaction is switched off.

(2) The (anti)commutator of Q with any observable is equal to zero.

(3) The operator Q has a ghost number equal to 1.

(4) Q is nilpotent,

$$[Q,Q]_+=2Q^2=0\,. \tag{4}$$

(5) For any linear combinations $\tilde{c}_i^\alpha=\sum_j b_{ij}(k)\times c_j^\alpha(k)$, $\tilde{\bar{c}}_i^\alpha=\sum_j b_{ij}(k)\bar{c}_j^\alpha(k)$ either $[Q_0, \tilde{c}_j^\alpha]_\pm\neq 0$, or $[Q_0, \tilde{\bar{c}}_j^\alpha]_\pm\neq 0$.

(6) The operators $a_i^*(k)$ (anti)commuting with Q_0 are physical, that is the states generated by these operators have a positive norm.

If the operator Q satisfies the conditions (1)–(6), the following theorem is valid:

The S-matrix constructed with the help of the lagrangian is unitary in the subspace (1) and the corresponding asymptotic states have a non-negative norm.

First of all let us note that for asymptotic states condition (1) is simplified:

$$Q|\phi\rangle = 0 \to Q_0|\phi_{as}\rangle = 0 . \tag{5}$$

Here we have used the assumption about linearization of the asymptotical dynamics which is valid in our case at least in the framework of perturbation theory.

One can easily verify that the operator Q_0 commutes with the S-matrix [7] and therefore the S-matrix is unitary in the subspace of asymptotic states satisfying eq. (5). Now we shall prove that this subspace has a non-negative norm.

The most general form of the operator Q_0 which is quadratic in the fields and has a ghost number equal to 1 is

$$Q_0 = \int \mathrm{d}k \left(\sum_{i=1}^{N_1} \sum_{j=1}^{M_1} a_i^*(k) \beta_1^{ij}(k) c_j^1(k) + \sum_{\alpha=2} \sum_{j=1}^{N_2} \sum_{l=1}^{M_2} c_j^{*\alpha-1}(k) \beta_\alpha^{jl}(k) c_l^\alpha(k) + \text{h.c.} \right) . \tag{6}$$

Let us introduce linear dependent fields $\tilde{c}_i^\alpha = \sum_j \beta_\alpha^{ij} c_j^\alpha$. If the maximal number of their independent components is n_α, then the fields $\tilde{c}_i^\alpha$ may be represented as follows:

$$\tilde{c}_i^\alpha = \{ c_j^{\alpha-}, \tilde{c}_d^{\alpha-} \} , \quad i = 1, ..., N_\alpha ,$$
$$j = 1, ..., n_\alpha , \quad d = n_\alpha + 1, ..., N_\alpha , \tag{7}$$

where

$$\tilde{c}_d^{\alpha-} = \sum_{j=1}^{n_\alpha} b_{dj}^\alpha c_j^{\alpha-} . \tag{8}$$

In terms of these fields the operator Q_0 looks as follows:

$$Q_0 = \int \mathrm{d}k \left(\sum_{j=1}^{n_1} (a_j^* + a_d^* b_{dj}) c_j^{1-} + \sum_{\alpha=2} \sum_{j=1}^{n_\alpha} (c_j^{*\alpha-1} + \tilde{c}_d^{*\alpha-1} b_{dj}^\alpha) c_j^{\alpha-} + \text{h.c.} \right) . \tag{9}$$

It follows from the nilpotency condition (4) that

$$[a_j + b_{dj}^{1*} a_d, a_k^* + a_c^* b_{ck}^1] = 0 , \tag{10}$$

$$[c_j^{1-}, \tilde{c}_k^{*1} + \tilde{c}_d^{*1} b_{dk}^2]_+ = 0 , \tag{11}$$

etc. Here for definiteness we assumed that the fields a_i have Bose statistics.

Eq. (10) leads to the following commutation relations for the fields:

$$a_i^+ = \frac{a_i + b_{di}^{1*} a_d}{\sqrt{2}} , \quad a_j^- = \frac{a_i - b_{di}^{1*} a_d}{\sqrt{2}} g_{ij} , \tag{12}$$

$$[a_i^+, (a_j^+)^*] = [a_i^-, (a_j^-)^*] = 0 ,$$
$$[a_i^+(k), (a_j^-)^*(k')] = \delta_{ij} \delta(k - k') . \tag{13}$$

The first term in the equation for Q_0 may be written in the form

$$Q_0^0 = \int \mathrm{d}k \left(\sum_{i=1}^{n_1} [(a_i^+)^* c_i^- - (c_i^{1-})^* a_i^+] \right) . \tag{14}$$

If n_1 is equal to the number of the original operators then due to the independence of c_j^{1-}, the operator Q_0^0 acts non-trivially on all the ghost fields $\tilde{c}_i^1(k)$. The matrix β_1^{ij} has an inverse

$$\beta_1^{ij} g_1^{jk} (\beta_1^{-1})^{kl} = \delta^{il} , \tag{15}$$

and one can introduce creation operators

$$(\tilde{c}_k^{1+})^* = (\beta_1^{-1})_{kl} (\tilde{c}_l^1)^* \tag{16}$$

such that

$$[c_i^{1-}(k), (\tilde{c}_j^{1+})^*(k')]_+ = \delta_{ij} \delta(k - k') . \tag{17}$$

The operator Q_0^0 acts in the space spanned by the vectors generated by the operators $(a_i^+)^*$, $(a_j^-)^*$, $(c_i^{1-})^*$, $(\tilde{c}_k^{1+})^*$ and satisfies the conditions (1)–(6). In this case only one generation of ghosts is needed. This situation is realized in the Yang–Mills theory and in other models with irreducible constraints and closed algebra.

If the number of independent operators is less than the original number of ghosts ($n_1 < M_1$) the operator Q_0^0 does not act on some of the fields $\tilde{c}_i^1$ and to construct the operator satisfying the conditions (1)–(6) one needs higher ghost generations.

The operator Q_0^0 is constructed in the same way as before, the creation operators $(\tilde{c}_k^{1+})^*$ being defined by eq. (16) where $(\beta_1^{-1})_{kl}$ is a right inverse matrix. The next term in Q_0 looks as follows:

$$Q_0^1=\int\left(\sum_{i=1}^{n_2}(\bar{c}_i^{1*}+\bar{c}_d^{1*}b_{di}^2)c_i^{2-}+\text{h.c.}\right)\mathrm{d}\boldsymbol{k}\,. \tag{18}$$

The operator Q_0^1 annihilates all the vectors created by $(a_i^+)^*$, $(a_j^-)^*$, $(c_k^{1-})^*$, $(\bar{c}_l^{1+})^*$ due to condition (II). The operators

$$\hat{c}_i^{1+}=\bar{c}_i^1+b_{id}^{2*}\bar{c}_d^1\equiv\gamma_{ij}\bar{c}_j^1 \tag{19}$$

are independent and one may introduce creation operators

$$(\hat{c}_i^{1-})^*=c_k^{1*}(\gamma^{-1})_{ki}\,,$$

$$[\hat{c}_i^{1+}(\boldsymbol{k}),(\hat{c}_j^{1-})^*(\boldsymbol{k}')]_+=\delta_{ij}\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{20}$$

Defining in the same way

$$(\bar{c}_k^{2+})^*=\bar{c}_l^{2*}(\beta_2^{-1})_{lk}\,, \tag{21}$$

we get the following representation for the Q_0^1:

$$Q_0^1=\int\mathrm{d}\boldsymbol{k}\left(\sum_{i=1}^{n_2}(\hat{c}_i^{1+})^*c_i^{2-}+(c_i^{2-})^*\hat{c}_i^{1+}\right). \tag{22}$$

This operator acts in the space spanned by the operators $(\hat{c}_i^{1+})^*$, $(\hat{c}_k^{1-})^*$, $(c_j^{2-})^*$, $(\bar{c}_l^{2+})^*$. Again if n_2 is equal to the number of the original operators c_i^2 then the series in (9) terminates, otherwise we must consider the next term, etc.

Therefore the general structure of the operator Q_0 is given by the following expression:

$$Q_0=\int\mathrm{d}\boldsymbol{k}\left(\sum_{i=1}^{n_1}(a_i^+)^*c_i^{1-}+\sum_{i=1}^{n_2}(\hat{c}_i^{1+})^*c_i^{2-}\right.$$

$$\left.+\sum_{i=1}^{n_3}(\hat{c}_i^{2+})^*c_i^{3-}+\ldots+\text{h.c.}\right), \tag{23}$$

where each term operates in a separate subspace.

Now we can study the problem of positivity of the physical states norm. It is convenient to change notations. In the following we shall denote all Bose operators as a_α, $\bar{a}_\alpha$,

$$[a_\alpha(\boldsymbol{k}),\bar{a}_\beta^*(\boldsymbol{k}')]=\delta_{\alpha\beta}\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \tag{24}$$

and all Fermi operators as c_α, $\bar{c}_\alpha$,

$$[c_\alpha(\boldsymbol{k}),\bar{c}_\beta^*(\boldsymbol{k}')]_+=\delta_{\alpha\beta}\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{25}$$

In these notations the operator Q_0 looks as follows:

$$Q_0=\int\mathrm{d}\boldsymbol{k}\sum_{\alpha=1}^{N}[\bar{a}_\alpha^*(\boldsymbol{k})c_\alpha(\boldsymbol{k})+c_\alpha^*(\boldsymbol{k})\bar{a}_\alpha(\boldsymbol{k})]\,, \tag{26}$$

where N is the total number of different types of particles. In particular N may be equal to ∞.

An arbitrary state vector may be represented as a sum of the vectors

$$|\phi\rangle=|\phi\rangle_{\text{ph}}\otimes|\tilde{\phi}\rangle\,, \tag{27}$$

where $|\phi\rangle_{\text{ph}}$ is a positive norm state, and the vector $|\tilde{\phi}\rangle$ looks as follows:

$$|\tilde{\phi}\rangle=\sum\int\prod_{\alpha=1}^{N}\bar{a}_\alpha^*(\boldsymbol{p}_1^\alpha)\ldots\bar{a}_\alpha^*(\boldsymbol{p}_{n_\alpha}^\alpha)a_\alpha^*(\boldsymbol{q}_1^\alpha)\ldots a_\alpha^*(\boldsymbol{q}_{m_\alpha}^\alpha)$$

$$\times c_\alpha^*(\boldsymbol{s}_1^\alpha)\ldots c_\alpha^*(\boldsymbol{s}_{k_\alpha}^\alpha)\bar{c}_\alpha^*(\boldsymbol{t}_1^\alpha)\ldots\bar{c}_\alpha^*(\boldsymbol{t}_{l_\alpha}^\alpha)$$

$$\times G_{k_1\ldots k_N,l_1\ldots l_N}^{n_1\ldots n_N,m_1\ldots m_N}(\boldsymbol{p}_1^1\ldots\boldsymbol{p}_{n_N}^N,\boldsymbol{q}_1^1\ldots\boldsymbol{q}_{m_N}^N,\boldsymbol{s}_1^1\ldots\boldsymbol{s}_{k_N}^N,\boldsymbol{t}_1^1\ldots\boldsymbol{t}_{l_N}^N)$$

$$\times\mathrm{d}\boldsymbol{p}_1^1\ldots\mathrm{d}\boldsymbol{t}_{l_N}^N\,, \tag{28}$$

where the functions G are symmetric with respect to the arguments p_i^α, q_j^α for fixed α, and antisymmetric with respect to s_i^α, t_j^α.

Substituting the vector (27) into eq. (1) we get the following relation between the functions G:

$$l_\beta G_{k_1\ldots k_\beta\ldots k_N,l_1\ldots l_\beta\ldots l_N}^{n_1\ldots n_\beta\ldots n_N,m_1\ldots m_\beta\ldots m_N}(\boldsymbol{p}_1^1\ldots\boldsymbol{p}_{n_N}^N,$$

$$\boldsymbol{q}_1^1\ldots\boldsymbol{q}_{m_N}^N,\boldsymbol{s}_1^1\ldots\boldsymbol{s}_{k_N}^N,\boldsymbol{t}_1^1\ldots\boldsymbol{t}_{l_N}^N)$$

$$+(m_\beta+1)(-1)^{\sum_{\alpha=\beta}^N k_\alpha+\sum_{\alpha=1}^{\beta-1}l_\alpha}$$

$$\times G_{k_1\ldots k_\beta-1\ldots k_N,l_1\ldots l_\beta-1\ldots l_N}^{n_1\ldots n_\beta+1\ldots n_N,m_1\ldots m_\beta+1\ldots m_N}(\boldsymbol{p}_1^1\ldots\boldsymbol{p}_{n_{\beta-1}}^{\beta-1}\boldsymbol{t}_1^\beta\boldsymbol{p}_1^\beta\ldots\boldsymbol{p}_{n_N}^N,$$

$$\boldsymbol{q}_1^1\ldots\boldsymbol{q}_{m_{\beta-1}}^{\beta-1}\boldsymbol{s}_1^\beta\boldsymbol{q}_1^\beta\ldots\boldsymbol{q}_{m_N}^N,\boldsymbol{s}_1^1\ldots\boldsymbol{s}_{k_{\beta-1}}^{\beta-1}\boldsymbol{s}_2^\beta\ldots\boldsymbol{s}_{k_N}^N,$$

$$\boldsymbol{t}_1^1\ldots\boldsymbol{t}_{l_{\beta-1}}^{\beta-1}\boldsymbol{t}_2^\beta\ldots\boldsymbol{t}_{l_N}^N)=0\quad(m_\beta,l_\beta\neq0)\,,$$

$$G_{k_1\ldots k_N,l_1\ldots l_N}^{n_1\ldots0\ldots n_N,m_1\ldots m_N}=G_{k_1\ldots0\ldots k_N,l_1\ldots l_N}^{n_1\ldots n_N,m_1\ldots m_N}=0\,. \tag{29}$$

In this equation it is understood that the functions G are symmetrized according to the corresponding Young tableaux which can be easily deduced from the representation (28).

If we now calculate the contribution to the norm of the terms with occupation numbers $(n_1\ldots n_\beta\ldots n_N, m_1\ldots m_\beta\ldots m_N, k_1\ldots k_\beta\ldots k_N, l_1\ldots l_\beta\ldots l_N)$ and $(n_1\ldots n_\beta+1\ldots n_N, m_1\ldots m_\beta+1\ldots m_N, k_1\ldots k_\beta-1\ldots k_N, l_1\ldots l_\beta-1\ldots l_N)$ and take into account relation (29) we find that these terms have the same absolute value but differ by a sign. Therefore they cancel in the sum and only the terms with zero occupation numbers give a non-zero contribution.

An arbitrary state vector satisfying eq. (1) may be represented in the form

$$|\phi\rangle = |\phi\rangle_{ph} \oplus |\phi_0\rangle\,, \qquad (30)$$

where $|\phi\rangle_{ph}$ has a positive norm and $|\phi_0\rangle$ has a zero norm. The vectors $|\phi_0\rangle$ are orthogonal to all other vectors. Hence we may identify all the vectors differing by $|\phi_0\rangle$. The positive norm physical space is the subspace (1) factorized with respect to the null vector space $\mathscr{H}_0$.

So we proved that if the theory satisfies the conditions (1)–(6) the S-matrix is unitary in the positive norm subspace and physical unitarity holds. No assumptions about the gauge group were needed. The only important limitation is the hypothesis of asymptotic dynamics linearization. It is valid in standard field theory at least in the framework of perturbation theory. However, in some models like first quantized strings it may not be true. The conditions (1)–(6) are met in a natural way in the BRST approach to gauge theories. Some examples of this procedure and a more detailed discussion will be given elsewhere.

References

[1] T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. 66 (1979) 1.
[2] G. Curci and R. Ferrary, Nuovo Cimento 35A (1976) 273.
[3] I.A. Batalin and G. Vilkovisky, Phys. Rev. D 28 (1983) 2567.
[4] I.A. Batalin and E.S. Fradkin, Phys. Lett. B 122 (1983) 157.
[5] M. Henneaux, Phys. Rep. 126 (1985) 1.
[6] R. Marnelius, Phys. Lett. B 99 (1981) 467.
[7] L.D. Faddeev and A.A. Slavnov, Gauge theories; introduction to quantum theory (Benjamin, New York, 1980).